Delayed Oscillation Phenomena in the FitzHugh Nagumo Equation

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We study the problem of the slow passage through a Hopf bifurcation point for the FitzHugh Nagumo equation (FHN)

$$v_t = Dv_{xx} - f(v) - w + I_0 + \varepsilon t, \tag{0.1a}$$

$$w_t = bv - b\gamma w, \tag{0.1b}$$

where f has some properties so that the system has a Hopf bifurcation at $I = I_-$ when $\varepsilon = 0$ and $I = I_0 + \varepsilon t$ is regarded as a parameter independent of t. The experimental results of Jakobsson and Guttman showed that large amplitude oscillations occurred only after I reaches a value well above I_- when ε is positive and small. The paper of S. M. Baer, T. Erneux, and J. Rinzel (*Siam J. Appl. Math.* **49**, 1989, 55–71) studied these phenomena numerically, and produced a prediction of the ignition (jumping) time for the system. In this work, we provide a rigorous proof of the results conjectured by Baer, Erneux, and Rinzel (referenced above). We show that if we start the solution of (0.1) at any point near the frame solution, which is the zero of the right-hand side of (0.1), at any $I_i < I_-$, then the solution stays near the frame solution until I reaches some $I_q > I_-$. Furthermore, for those cases in which I_i is close to I_-, we show that the solution moves from the frame solution to become a large amplitude solution after I moves above some $I_q > I_-$.

1. INTRODUCTION OF THE PROBLEM

In 1952, Hodgkin and Huxley [1] suggested a model (HH) that describes the generation and propagation of a nerve impulse along the giant axon of a squid. In the early 1960s, FitzHugh [2] and Nagumo [3] provided a simpler model which contains the main qualitative features of the HH model. It is known as the FitzHugh Nagumo equation (FHN), and has the form

$$v_t = Dv_{xx} - f(v) - w + I(x, t), \tag{1.1a}$$

$$w_t = bv - b\gamma w, \tag{1.1b}$$

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where $f(v) = v(v-1)(v-a)$, D, b, and γ are positive constants, and $0 < a < 1/2$.

In the FHN model, v denotes the potential difference across the membrane of the axon; w represents a recovery current which is often taken to be the sum of all ion flows. I is an applied (or injected) electric current on the membrane, a control parameter of the experiments. Equation (1.1a) expresses Kirchoff's current law applied to the membrane; (1.1b) relates the recovery current with the potential. In addition, biophysical measurements show that

$$\gamma_1 \equiv \frac{1}{\gamma} - \frac{1-a+a^2}{3} > 0, \qquad \text{and} \qquad \gamma_2 \equiv \frac{1-a+a^2}{3} - b\gamma > 0 \tag{A}$$

if the membrane is in a low calcium environment. See [6–8].

Up to now, most of the experiments have been performed on the clamped squid axon which means that I and (v, w) do not depend on x. Hence the discussions were focused on

$$v_t = -f(v) - w + I, \tag{1.2a}$$

$$w_t = bv - b\gamma w. \tag{1.2b}$$

If the current I is kept constant during the process, a steady state solution of the system is determined by the equations

$$-f(v) - w + I = 0, \tag{1.3a}$$

$$bv - b\gamma w = 0. \tag{1.3b}$$

It has been shown [7, 8] for the existence of a unique solution $(V_0(I), W_0(I))$ to (1.3). Moreover, its components $V_0(I)$ and $W_0(I)$ increase as I increases. We define $(V_0(I), W_0(I))$ as the *frame solution* of the system (1.2).

When we consider the linearized stability problem

$$\frac{d\tilde{V}}{dt} = -f'(V_0(I))\,\tilde{V} - \tilde{W}, \tag{1.4a}$$

$$\frac{d\tilde{W}}{dt} = b\tilde{V} - b\gamma\tilde{W}, \tag{1.4b}$$

for $(V_0(I), W_0(I))$, we see that the stability of (1.4) is determined by the two eigenvalues of the Jacobian

$$\begin{pmatrix} -f'(V_0(I)) & -1 \\ b & -b\gamma \end{pmatrix}.$$

We see that there exist I_- and I_+ where $I_- < \frac{1}{3}(a+1) < I_+$ such that whenever $I < I_-$ or $I > I_+$, the eigenvalues are either complex with negative real parts, or real and negative, i.e., (1.4) is stable; whenever $I_- < I < I_+$, both eigenvalues have positive real parts, i.e., (1.4) is unstable. Details can be found in [7].

In the 1970s Cole [4] first discovered the two stable patterns of the system called "Accommodation" and "Repetitive firing." These are stable steady solutions and stable periodic solutions, respectively. Based on his indication, Troy [5] proved the fact that a Hopf bifurcation appears as I increases through I_- and as I decreases through I_+. The resulting periodic solutions are stable.

The production of an oscillating solution during the slow passage of a parameter through a critical value has been extensively studied in recent years. In such problems, the current $I(t)$ has the form of $I(t) = I_i + \varepsilon t$, where $0 < \varepsilon \ll 1$ is a very small parameter, called the ramp speed. Also $I_i < I_-$.

Since $I(t)$ is an increasing function of t, we write $t = (I - I_i)/\varepsilon$ and use I as the independent variable. Then the system (1.2) becomes

$$\varepsilon v_I = -f(v) - w + I, \tag{1.5a}$$

$$\varepsilon w_I = bv - b\gamma w. \tag{1.5b}$$

One might expect from the classical stability theory that the solution of (1.5) with the initial conditions

$$v(I_i) = V_0(I_i), \qquad w(I_i) = W_0(I_i)$$

stays close to $(V_0(I), W_0(I))$ until I reaches the value I_-, and then jumps away from the frame solution shortly after I increases past I_-. This seems to be reasonable since the frame solution $(V_0(I), W_0(I))$ loses its stability there.

Various recent experiments and computer approximations have indicated that this classical argument is not true in general. In 1981, Jakobsson and Guttman [6] found the "Reversed Accommodation" phenomenon in which large amplitude oscillations occurred only after I reaches a value well above I_-. Furthermore, their laboratory results showed that this delay depends on the initial current $I_i < I_-$. This property may be of importance in the study of threshold properties of nervous systems.

In 1988, Baer, Erneux, and Rinzel [7, 8] did an extensive computational study of the FHN model for the delayed oscillation phenomena, and they began to consider the corresponding mathematical problem. They produced a very interesting prediction of the ignition (jumping) time for the system which was later shown to be correct.

Meanwhile, the problem of the slow passage through the critical value was studied independently by Šiškova [9] for a particular model. Later a more general case of stability persistence was presented by Neistadt [10–12]. In his papers, he attempted to provide a non-oscillation condition for a singularly perturbed ODE system. However, his proof involved a delay equation of $u(z)$ in the complex domain where the delay term contained $u(\bar{z})$. His argument needs some further treatment, such as the ones in [17], in order to obtain the claimed results. Arnold [13] has pointed out the importance of the study of such problems.

In this work we provide a rigorous proof of the results conjectured in [7, 8]. We show that if we start the solution (1.5) at any point near the frame solution $(V_0(I), W_0(I))$ at $I_i < I_-$, then the solution stays near $(V_0(I), W_0(I))$ until I reaches some $I_q > I_-$. Furthermore, for those cases in which I_i is close to I_-, a description of how the solution moves from the frame solution to become a large amplitude solution after $I > I_q$ is given. In another related paper [17], we extend these results to the non-spatially uniform case where the FHN model is a partial differential equation system of the form (1.1). [18] provided some more details of this paper and [17].

2. Some Transformations

Since we are interested in the difference between the solutions of (1.5) and the frame solution, we use the new dependent variables $V = v - V_0$, $W = w - W_0$. The system becomes

$$\varepsilon V_I = f'(V_0(I))\, V - W - \tfrac{1}{2} f''(V_0(I))\, V^2 - \tfrac{1}{6} V^3 - \varepsilon V_0'(I), \tag{2.1a}$$

$$\varepsilon W_I = bV - b\gamma W - \varepsilon W_0'(I). \tag{2.1b}$$

Since $V_0'(I) > 0$, we can introduce the new independent variable $J = V_0(I) - \frac{1}{3}(a+1)$. We obtain the system

$$\varepsilon V_J = (3J^2 + \gamma_1)((-3J^2 + 3J_-^2 + b\gamma)\, V - W - 3JV^2 - V^3) - \varepsilon, \tag{2.2a}$$

$$\varepsilon W_J = (3J^2 + \gamma_1)(bV - b\gamma W) - \frac{1}{\gamma}\varepsilon, \tag{2.2b}$$

where $J_- = V_0(I_-) - \frac{1}{3}(a+1) = -\frac{1}{3}\sqrt{a^2 - a + 1 - 3b\gamma} = -\frac{1}{3}\sqrt{3\gamma_2}$ and γ_1, γ_2 are defined in (A).

All the above-mentioned properties are preserved under this change of variables. The critical values of J at which the system (2.2) changes its stability are J_- defined above and $J_+ = -J_- > 0$. The two eigenvalues of the linearization about $V = W = 0$ become

$$\lambda_i(J) = \tfrac{1}{2}(3J^2 + \gamma_1)(-3J^2 + 3J_-^2 - (-1)^i \sqrt{(3J^2 - 3J_-^2)^2 - 4b\gamma(3J^2 + \gamma_1)}). \tag{2.3}$$

We see that $\operatorname{Re} \lambda_i(J) \leqslant 0$ when $J \leqslant J_-$ or $J \geqslant J_+$, while $\operatorname{Re} \lambda_i(J) > 0$ when $J_- < J < J_+$. Also $\operatorname{Im} \lambda_i(J_-) = (-1)^{i-1} k \neq 0$ and $\operatorname{Im} \lambda_i(J_+) = (-1)^{i-1} k \neq 0$ where $k = |(3J^2 - 3J_-^2)^2 - 4b\gamma(3J^2 + \gamma_1)|^{1/2} > 0$. (Note that $J_+ = V_0(I_+) - \frac{1}{3}(a+1)$.)

We observe from (2.3) that $\lambda_i(J)$ is symmetric with respect to 0.

As we study (2.3) more carefully, we find that near the two bifurcation points J_- and $J_+ = -J_-$, these eigenvalues are complex. Through direct calculations, we know that there exist $J_T^1 < J_-$ and $J_T^4 = -J_T^1 > J_+$ at which the two eigenvalues coincide, and the eigenvalues $\lambda_i(J)$ become real when $J < J_T^1$ and $J > J_T^4$. These real points where the eigenvalues coincide are named *Turning Points*. For some ranges of parameters a, b, γ, other turning points may appear at $J_- < J_T^2 \leqslant 0 \leqslant J_T^3 = -J_T^2 \leqslant J_+$, so that the eigenvalues are real for $J_T^2 \leqslant J \leqslant J_T^3$. A more detailed discussion of the various cases will be given in Section 6.

3. Solutions before the Critical Point J_-

We first study the behavior of the solution of (2.2) when $J < J_-$, that is, before J reaches the first critical value J_-. We assume that the initial conditions at $J_0 < J_-$ have the property that

$$(|V(J_0)|^2 + |W(J_0|^2)^{1/2} \leqslant M_1 \varepsilon, \tag{3.1}$$

for some constant M_1.

THEOREM 1. *Let $(V, W) = (V(J, \varepsilon), W(J, \varepsilon))$ be a family of solutions of the system (2.2) depending on the parameter ε with initial conditions at $J = J_0 \leqslant J_-$ which satisfy (3.1). Then there exist $M_2 = M_2(M_1)$ and $\varepsilon_0 = \varepsilon_0(M_1) > 0$ such that when $0 < \varepsilon < \varepsilon_0$,*

$$(|V(J)|^2 + |W(J)|^2)^{1/2} \leqslant M_2 \varepsilon, \tag{3.2}$$

for $J_0 \leqslant J \leqslant J_-$.

Proof. We prove the most complicated case in which $J_0 \leqslant J_T^1$. As we shall see, other cases such as $J_T^1 \leqslant J_0 \leqslant J_-$ can be taken care of by applying a subset of the steps in this proof. The idea here is similar to ones in [15–16] except for the dealing of J_T^1.

We construct an ε-neighborhood of the origin and then show that the solution never gets out of this ε-neighborhood. We define $J_{\max}$ (the first exit time) by

$$J_{\max} = \inf\{J \in [J_0, J_-) \mid (|V(J)|^2 + |W(J)|^2)^{1/2} \geqslant M_2 \varepsilon\}. \tag{3.3}$$

If $M_2 > M_1$, then $J_{\max} > J_0$. To get Theorem 1, we aim to show that $J_{\max} = J_-$.

We let $(V_1(J), W_1(J))$ be the solution of the linear system

$$(3J^2 + \gamma_1)\begin{pmatrix} -3J^2 + 3J_-^2 + b\gamma & -1 \\ -b & -b\gamma \end{pmatrix}\begin{pmatrix} V_1(J) \\ W_1(J) \end{pmatrix} + \begin{pmatrix} -1 \\ -1/\gamma \end{pmatrix} = 0. \tag{3.4}$$

It is easy to show that $\binom{V_1(J)}{W_1(J)}$ and $\binom{V_1'(J)}{W_1'(J)}$ are bounded.

We now change to the new variables

$$\hat{V} = V - \varepsilon V_1, \tag{3.5a}$$

$$\hat{W} = W - \varepsilon W_1. \tag{3.5b}$$

$\binom{\hat{V}(J)}{\hat{W}(J)}$ satisfies the system

$$\varepsilon\begin{pmatrix} \hat{V}_J \\ \hat{W}_J \end{pmatrix} = Q_0\begin{pmatrix} \hat{V} \\ \hat{W} \end{pmatrix} + \begin{pmatrix} F_1(\hat{V}, J) + G_1(J) \\ G_2(J) \end{pmatrix}, \tag{3.6}$$

where

$$Q_0 = (3J^2 + \gamma_1)\begin{pmatrix} -3J^2 + 3J_-^2 + b\gamma & -1 \\ -b & -b\gamma \end{pmatrix},$$

$|F_1(\hat{V}, J)| \leqslant \varepsilon k_1 |\hat{V}| + k_2 |\hat{V}|^2 + k_3 |\hat{V}|^3$, and $|G_1(J)|$, $|G_2(J)| \leqslant \varepsilon^2 k_4$ since $V_1(J)$, $V_1'(J)$, $W_1(J)$, $W_1'(J) = O(1)$.

The initial conditions have the property that

$$(|\hat{V}(J_0)|^2 + |\hat{W}(J_0)|^2)^{1/2} \leqslant (M_1 + \max_{J_0 \leqslant J \leqslant J_-} (|V_1|^2 + |W_1|^2)^{1/2})\,\varepsilon.$$

For $J_0 < J < J_{\max}$, the nonlinear terms are estimated as

$$\begin{aligned} |F_1(\hat{V}, J)| &\leqslant \varepsilon k_1 |\hat{V}| + k_2 |\hat{V}|^2 + k_3 |\hat{V}|^3, \\ &\leqslant k_0 \varepsilon^2 + 2k_1 M_2 \varepsilon^2 + 4k_2 M_2^2 e^2 + 8k_3 M_2^3 \varepsilon^3, \\ |G_1(J)| &\leqslant \varepsilon^2 k_4, \\ |G_2(J)| &\leqslant \varepsilon^2 k_4, \end{aligned} \tag{3.7}$$

where k_i are constants independent of ε.

We now define $J_{\varepsilon^{1/2}} < J_-$ to be the value of J such that $\mathrm{Re}(\lambda_i(J)) = -\varepsilon^{1/2}$ for sufficiently small ε. We observe that

$$\frac{d}{dJ}(\mathrm{Re}\,\lambda_i(J))|_{J=J_-} > 0. \tag{3.8}$$

Therefore $\mathrm{Re}\,\lambda_i(J)$ increases to 0 as J increases to J_-, $J_{\varepsilon^{1/2}}$ is well defined and unique, and

$$|J_- - J_{\varepsilon^{1/2}}| \leqslant K\varepsilon^{1/2} \tag{3.9}$$

when $\varepsilon < \varepsilon_0$ is sufficiently small.

Since $\mathrm{Re}\,\lambda_i(J) < 0$ when $J_0 \leqslant J \leqslant J_-$, we see if ε is small enough, then $\mathrm{Re}\,\lambda_i(J) \leqslant -\varepsilon^{1/2}$ for $J \leqslant J_{\varepsilon^{1/2}}$.

We consider the system (3.6) in 5 intervals:

$$\begin{aligned}
I_1 &= [J_0, J_T^1 - c\varepsilon^{1/2}];\\
I_2 &= [J_T^1 - c\varepsilon^{1/2}, J_T^1 + c\varepsilon^{1/2}];\\
I_3 &= [J_T^1 + c\varepsilon^{1/2}, \tfrac{1}{2}(J_T^1 + J_-)];\\
I_4 &= [\tfrac{1}{2}(J_T^1 + J_-), J_{\varepsilon^{1/2}}];\\
I_5 &= [J_{\varepsilon^{1/2}}, J_-].
\end{aligned} \tag{3.10}$$

We will transform the Q_0 into a canonical form which varies in different I_j.

(A) When $J \in I_1 = [J_0, J_T^1 - c\varepsilon^{1/2}]$, the eigenvalues $\lambda_1(J)$, $\lambda_2(J)$ of the matrix Q_0 are two negative real numbers.

If $A_1(J) = (\xi_1(J), \xi_2(J))$ where ξ_1, ξ_2 are the eigenvectors of Q_0 corresponding to λ_1, λ_2, then $\|A_1(J)\| + \|(d/dJ)\,A_1(J)\| \leqslant k_6$ where $\|A\|$ is a norm for the matrix A. Since J is kept away from J_T^1 where eigenvectors collapse at a distance of $c\varepsilon^{1/2}$, $\|A_1^{-1}\| \leqslant k\varepsilon^{-1/2}$ when $J \in I_1$.

We define the new variable vector $\binom{U_1}{U_2}$ by

$$\begin{pmatrix}\hat{V}\\ \hat{W}\end{pmatrix} = A_1(J)\begin{pmatrix}U_1\\ U_2\end{pmatrix}.$$

Thus, we can simply write

$$\varepsilon\frac{d}{dJ}\begin{pmatrix}U_1\\ U_2\end{pmatrix} = \begin{pmatrix}\lambda_1(J) & 0\\ 0 & \lambda_2(J)\end{pmatrix}\begin{pmatrix}U_1\\ U_2\end{pmatrix} + \varepsilon^{-1/2}\begin{pmatrix}F_3\\ F_4\end{pmatrix}, \tag{3.11}$$

where $|F_3|$, $|F_4| \leqslant h(M_2, (V_1, W_1))\,\varepsilon^2$ when $J < J_{\max}$.

$\hat{U} = |(U_1, U_2)|$ satisfies

$$\varepsilon \frac{d}{dJ} \hat{U} \leqslant \mathrm{Re}(\lambda_1(J))\, \hat{U} + \varepsilon^{3/2} h(M_2) \tag{3.12}$$

for $J \leqslant J_{\max}$.

We observe that $\lambda_i(J) < -\alpha_a < 0$ when $J \in I_1$.

By a direct integration,

$$\hat{U}(J) \leqslant K_1(M_1 + \max |(V_1, W_1)|)\, \varepsilon e^{-(1/\varepsilon)\alpha_a(J - J_0)} + h(M_2, (V_1, W_1))\, \varepsilon^{3/2}, \tag{3.13}$$

where the constant K_1 is caused by the transformation from the $(\hat{V}, \hat{W})$ to (U_1, U_2).

When $\varepsilon \leqslant \varepsilon_0(M_1)$ is small enough, and

$$M_2 = M_2^{\alpha}(M_1) = 4K_1^2(M_1 + \max |(V_1, W_1)|),$$
$$|(V, W)(J)| < M_2^{\alpha} \varepsilon \qquad \text{when} \quad J \in I_1.$$

(B) When $J \in I_2 = [J_T^1 - c\varepsilon^{1/2}, J_T^1 + c\varepsilon^{1/2}]$, $Q_0(J) = Q_0(J_T^1) + O(\varepsilon^{1/2})$. If we define (U_1, U_2) by

$$\begin{pmatrix} \hat{V} \\ \hat{W} \end{pmatrix} = \tilde{A}_1 \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}, \tag{3.14}$$

where $\tilde{A}$ is the matrix such that

$$Q_0(J_T^1)\, \tilde{A}_1 = \tilde{A}_1 \begin{pmatrix} \lambda_1(J_T^1) & 0 \\ \sigma_A & \lambda_2(J_T^1) \end{pmatrix},$$

then we can obtain

$$\varepsilon \frac{d}{dJ} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = \begin{pmatrix} \lambda_1(J_T^1) & 0 \\ \sigma_A & \lambda_2(J_T^1) \end{pmatrix} \begin{pmatrix} U_1(J) \\ U_2(J) \end{pmatrix}$$
$$+ \tilde{A}_1^{-1} O(\varepsilon^{1/2}) \begin{pmatrix} U_1(J) \\ U_2(J) \end{pmatrix} + \tilde{A}_1^{-1} \begin{pmatrix} F_1 + G_1 \\ G_2 \end{pmatrix}, \tag{3.15}$$

where σ_A can be chosen arbitrarily, and $\|\tilde{A}^{-1}\| = O(\sigma_A^{-1})$. Thus if we choose $\sigma_A = \varepsilon^{1/4}$, then the linear portion is positive definite, and the higher order portion is bounded by $\varepsilon^{1/4} K_2 |(U_1, U_2)|^2$ when ε is small. The result that $|(V, W)| \leqslant M_2^b \varepsilon = M_2^b(M_1)\varepsilon$ for $J \in I_2$ can be proved in the same manner as I_1.

(C) When $I \in I_3 \cup I_4 \cup I_5 = [J_T^1 + c\varepsilon^{1/2}, J_-]$, we notice that the eigenvalues $\lambda_1(J)$, $\lambda_2(J)$ are conjugate to each other, and $\xi_2(J) = \bar{\xi}(J)$.

We define the complex-valued new variable vector $\binom{U_1}{U_2}$ by

$$\begin{pmatrix} \hat{V} \\ \hat{W} \end{pmatrix} = A_1(J) \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}, \tag{3.16}$$

where A_1 is defined as before.

We see

$$\varepsilon \frac{d}{dJ} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = \begin{pmatrix} \lambda_1(J) & 0 \\ 0 & \lambda_2(J) \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} + \begin{pmatrix} F_5 \\ F_6 \end{pmatrix}.$$

This leads to the inequality

$$\varepsilon \frac{d}{dJ} \hat{U} \leqslant \mathrm{Re}(\lambda_1(J))\, \hat{U} + |F_7(\hat{U})|, \tag{3.17}$$

where $\hat{U} = |(U_1, U_2)|$ and F_7 contains higher order terms of (U_1, U_2).

It depends on the intervals to which J belongs that λ_1 and F_7 can be estimated individually.

When $J \in I_3$, $\mathrm{Re}\, \lambda_1(J) < -\alpha_c$ and $\|A_1^{-1}\| \leqslant K\varepsilon^{-1/2}$ so that $|F_7| \leqslant \varepsilon^{3/2} h(M_2)$. When $J \in I_4$, $|F_7| \leqslant h(M_2, |V_1|, |W_1|)\, \varepsilon^2$ and $\mathrm{Re}\, \lambda_1(J) \leqslant -\varepsilon^{1/2}$. When $J \in I_5$, $\mathrm{Re}(\lambda_1(J)) \leqslant 0$ $|F_7| \leqslant h(M_2, |V_1|, |W_1|)\, \varepsilon^2$, and also $|I_5| < K\varepsilon^{1/2}$.

From all the facts above, we are able to show that $|(V, W)| \leqslant M_2^c \varepsilon$ for $J \in I_3 \cup I_4 \cup I_5$, provided $M_2^c = M_2^c(M_1, V_1, W_1)$ is large enough independent of ε and $\varepsilon < \varepsilon_0(M)$.

Finally, let $M_2 = \max(M_2^a, M_2^b, M_2^c)$. Then $J_{\max} = J_-$. This completes the proof.

Remark 1. If we consider all the inequalities more carefully, we can obtain the expression

$$(V, W) = \varepsilon(V_1, W_1) + O(\varepsilon^{5/4}) + \varepsilon s(J_0, M_1), \tag{3.18}$$

where $|s(J_0, M_1)| \leqslant K M_1\, e^{-(1/\varepsilon)\alpha(J_0)(J - J_0)}$. Therefore when $J > J_0 + C\varepsilon\, |\ln \varepsilon|$,

$$\frac{1}{k} |(V_1, W_1)|\, \varepsilon \leqslant |(V, W)| \leqslant k\, |(V_1, W_1)|\, \varepsilon \tag{3.19}$$

provided C is large enough. In other words, $0 < c_1 \leqslant |(V(J), W(J))|/\varepsilon \leqslant c_2 < \infty$ when $J > J_0 + C\varepsilon\, |\ln \varepsilon|$.

We notice that even if the solution is not in the ε-neighborhood of the frame solution at J_0, the solution goes into the ε-neighborhood of the frame solution in a very short time. Therefore after the solution enters the ε-neighborhood, we are able to apply Theorem 1 to get the result as follows.

COROLLARY 1. *Let* $(V, W) = (V(J, \varepsilon), W(J, \varepsilon))$ *be a family of solutions of the system* (2.2) *depending on the parameter* ε *with initial conditions at* $J = J_0 \leqslant J_-$ *which satisfy*

$$(|V(J_0)|^2 + |W(J_0)|^2)^{1/2} \leqslant M_1 < M_1^0 \tag{3.20}$$

for some sufficiently small $M_1^0 > 0$. *Then there exist* $M_2 = M_2(M_1)$ *and* $\varepsilon_0 = \varepsilon_0(M_1) > 0$ *such that when* $0 < \varepsilon < \varepsilon_0$,

$$(|V(J)|^2 + |W(J)|^2)^{1/2} \leqslant M_2 \varepsilon$$

for $J_0 + K(M_1)\, \varepsilon\, |\ln \varepsilon| \leqslant J \leqslant J_-$.

Proof. We can show that if M_1^0 is sufficiently small, then

$$(V, W) = \varepsilon(V_1, W_1) + O(\varepsilon^{(5/4)}) + s(J_0, M_1),$$

where $|s(J_1, M_1)| \leqslant KM_1\, e^{-(1/\varepsilon)\alpha(J_0)(J - J_0)}$. Corollary 1 follows.

Remark. We see that solutions which start at J_0 outside the ε-neighborhood have no difference with the ones starting within the ε-neighborhood after $J > J_0 + K(M_1)\, \varepsilon\, |\ln \varepsilon|$. From now on, we just need to consider the solutions whose initial conditions satisfy (3.1). All results can be extended to the solutions whose initial conditions satisfy (3.20).

4. Basic Properties

Another natural corollary of Theorem 1 is that if we begin the solution from J^0 where $J_- < J^0 < J_+$, and solve the system backwards, then we see a similar phenomenon as in Theorem 1.

COROLLARY 2. *A family of solutions* $(V, W) = (V(J, \varepsilon), W(J, \varepsilon))$ *of* (2.2) *with initial conditions which satisfy*

$$(|V(J^0)|^2 + |W(J^0)|^2)^{1/2} \leqslant M_1 \varepsilon \tag{4.1}$$

at J^0 *where* $J_- < J^0 < J_+$ *satisfies* $|V(J)| + |W(J)| \leqslant M_2 \varepsilon$ *for* $J_- \leqslant J \leqslant J^0$ *provided* $\varepsilon \leqslant \varepsilon_0(M_1)$ *and* $M_2 = M_2(M_1)$.

Proof. We let $l = 2J_- - J$ be the new independent variable for the system. Then (2.2) becomes

$$\begin{aligned} -\varepsilon V_l = {} & (3(2J_- - l)^2 + \gamma_1)((-3(2J_- - l)^2 \\ & + 3J_-^2 + b\gamma)\, V - W - 3(2J_- - l)\, V^2 - V^3) - \varepsilon, \end{aligned} \tag{4.2a}$$

$$-\varepsilon W_l = (3(2J_- - l)^2 + \gamma_1)(bV - b\gamma W) - \frac{1}{\gamma}\varepsilon. \tag{4.2b}$$

The initial conditions satisfy

$$(|V(l^0)|^2+|W(l^0)|^2)^{1/2}\leqslant M_1\varepsilon,$$

where $l^0=2J_- - J^0<J_-$. We see that two eigenvalues of the Jacobian become $\sigma_1(l)=-\lambda_1(2J_- - l)$, $\sigma_2(l)=-\lambda_2(2J_- - l)$ with the real part $\mathrm{Re}(\sigma_1(l))=-\mathrm{Re}(\lambda(2J_- - l))<0$ for $l<J_-$. As $l\to(J_-)^-$, $\sigma(l)\to 0^-$. The rest of the proof including the treatment of the turning point is parallel to that of Theorem 1.

Another remarkable feature of the behavior of (V, W) is that the difference between two solutions within the ε-neighborhood has a very nice property provided ε is small enough. Here for simplicity, we exclude the turning points in our consideration.

Let (V_A, W_A) and (V_B, W_B) be two different solutions of (2.2).

We define

$$V_{AB}=V_A-V_B; \qquad W_{AB}=W_A-W_B.$$

THEOREM 2. *If (V_A, W_A) and (V_B, W_B) are two solutions of (2.2) on the interval $J_T^1\leqslant J_1\leqslant J\leqslant J_2<J_T^2$ and satisfy*

$$|(V_A(J), W_A(J))|\leqslant M_1\varepsilon, \qquad |(V_B(J), W_B(J))|\leqslant M_1\varepsilon,$$

then their difference (V_{AB}, W_{AB}) satisfies

$$\frac{1}{M_3}e^{(1/\varepsilon)\int_{J_1}^{J_2}\mathrm{Re}(\lambda_1(\tau))\,d\tau}\leqslant\frac{|(V_{AB}, W_{AB})(J_2)|}{|(V_{AB}, W_{AB})(J_1)|}\leqslant M_3\,e^{(1/\varepsilon)\int_{J_1}^{J_2}\mathrm{Re}(\lambda_1(\tau))\,d\tau}, \tag{4.3}$$

where $M_3=M_3(M_1)$ is a constant, and $\varepsilon<\varepsilon_0(M_1)$.

Proof. We look at

$$\varepsilon\frac{d}{dJ}\begin{pmatrix}V_{AB}\\ W_{AB}\end{pmatrix}=(3J^2+\gamma_1)\begin{pmatrix}-3J^2+3J_-^2+b\gamma+F_3'(\tilde V) & -1\\ b & -b\gamma\end{pmatrix}\begin{pmatrix}V_{AB}\\ W_{AB}\end{pmatrix}, \tag{4.4}$$

where $\tilde V$ is between V_A and V_B, $|F_3'(\tilde V)|\leqslant k_1\varepsilon+2k_2|\tilde V|+3k_3|\tilde V|^2$. Since $|V_A|\leqslant M_1\varepsilon$, and $|V_B|\leqslant M_1\varepsilon$, $|F_3'(\tilde V)|\leqslant k_6(M_1)\varepsilon$.

We may write

$$\begin{pmatrix}V_{AB}\\ W_{AB}\end{pmatrix}=A_1\begin{pmatrix}x\\ \bar x\end{pmatrix}. \tag{4.5}$$

We see

$$(\mathrm{Re}(\lambda_1(J))-K_1\varepsilon)\,|x|\leqslant\varepsilon\frac{d}{dJ}|x|\leqslant(\mathrm{Re}(\lambda_1(J))+K_1\varepsilon)\,|x|. \tag{4.6}$$

Inequality (4.3) follows from (4.6).

Remark. The significance of (4.3) is that once we know the difference at one point, we know it elsewhere, if the solutions stay near the frame solution.

5. STRUCTURES OF SOLUTIONS AT THE CRITICAL POINT J_-

We define $(V(J,s), W(J,s))$ to be the solution of (2.2) with the initial condition

$$V(s,s)=0; \qquad W(s,s)=0. \tag{5.1}$$

We now consider the curve $(V(J_-,s), W(J_-,s), J_-)$ in the plane $J=J_-$.

We note that $(V(J_-,J_-), W(J_-,J_-))=(0,0)$, and $(V(J_-,s), W(J_-,s))$ is a continuous curve depending on s.

We investigate the behavior of $(V(J_-,s), W(J_-,s))$ as s moves away from J_-.

We observe that if we can show

$$|(V(J_-,s_1), W(J_-,s_1))-(V(J_-,s_2), W(J_-,s_2))| \leqslant O(e^{-k/\varepsilon}), \qquad k>0$$

for some $s_1<J_-<s_2$, then (4.3) implies that $(V,W)(J,s_1)$ stays near $(V,W)(J,s_2)$ for $J_-\leqslant J\leqslant s_2$. Since $|(V(J,s_1), W(J,s_1))|=O(\varepsilon)$ for $s_1\leqslant J\leqslant J_-$ and $|(V(J,s_2), W(J,s_2))|=O(\varepsilon)$ for $J_-\leqslant J\leqslant s_2$, we can see that $|(V(J,s_1), W(J,s_1))|=O(\varepsilon)$ for $s_1\leqslant J\leqslant s_2$. This will prove that the solution does not jump away from the frame solution until after J passes $s_2>J_-$.

We first assume that $J_T^1<s_1<J_-<s_2<J_T^2$, so that there are no turning points between s_1 and s_2.

We let $V_J(J,s)$, $W_J(J,s)$ be the derivatives with respect to J, and $V_s(J,s)$, $W_s(J,s)$ the derivatives with respect to s.

Since $(V(J,s), W(J,s))$ satisfies the system (2.2), we have

$$\varepsilon\frac{d}{dJ}\begin{pmatrix} V_S \\ W_S \end{pmatrix}=(3J^2+\gamma_1)\begin{pmatrix} -3J^2+3J_-^2+b\gamma+F_3'(V,J) & -1 \\ b & -b\gamma \end{pmatrix}\begin{pmatrix} V_S \\ W_S \end{pmatrix}. \tag{5.2}$$

From $V(s,s)=W(s,s)=0$, we get

$$V_s(s,s)+V_J(s,s)=0; \qquad W_s(s,s)+W_J(s,s)=0.$$

By (2.2),

$$\varepsilon V_J(s, s) = (3s^2 + \gamma_1)((-3s^2 + 3J_-^2 + b\gamma)\, V(s, s) - W(s, s)) + F_3(V(s, s), s) - \varepsilon = -\varepsilon, \tag{5.3a}$$

$$\varepsilon W_J(s, s) = (3s^2 + \gamma_1)(bV(s, s) - b\gamma W(s, s)) - \frac{1}{\gamma}\varepsilon = -\frac{1}{\gamma}\varepsilon, \tag{5.3b}$$

so that

$$V_S(s, s) = 1; \qquad W_S(s, s) = \frac{1}{\gamma}. \tag{5.4}$$

Since $|V(J, s)| \leqslant K\varepsilon$ for all $s \leqslant J \leqslant J_-$ when $s < J_-$, and $|V(J, s)| \leqslant K\varepsilon$ for all $J_- \leqslant J \leqslant s$ when $s > J_-$ by Theorem 1 and Corollary 2, $|F_3'(V, J)| < K_1\varepsilon$ there. Since $\binom{V_S}{W_S}$ is real, we write

$$\begin{pmatrix} V_S \\ W_S \end{pmatrix} = A_1 \begin{pmatrix} Z \\ \bar{Z} \end{pmatrix}.$$

Then by (5.2), $(Z(J), \bar{Z}(J))$ satisfies the equation

$$\varepsilon \frac{d}{dJ}\begin{pmatrix} Z \\ \bar{Z} \end{pmatrix} = \begin{pmatrix} \lambda_1(J) & 0 \\ 0 & \bar{\lambda}_1(J) \end{pmatrix}\begin{pmatrix} Z \\ \bar{Z} \end{pmatrix} - \varepsilon A_1^{-1} \frac{dA_1}{dJ}\begin{pmatrix} Z \\ \bar{Z} \end{pmatrix} + A_1^{-1} H_2(V, J)\begin{pmatrix} Z \\ \bar{Z} \end{pmatrix}, \tag{5.5}$$

where H_2 is the matrix form of higher order terms. Since A_1 is nonsingular when J is not a turning point, $A_1^{-1}(J)$ is bounded.

Letting $Z(J) = R(J)\, e^{i\theta(J)}$, we obtain

$$\varepsilon \frac{d}{dJ} R = (\mathrm{Re}\lambda_1(J))\, R + f_1(J, \cos\theta, \sin\theta, \varepsilon)\, R \tag{5.6a}$$

$$\varepsilon \frac{d}{dJ}\theta = (I_m\lambda_1(J)) + f_2(J, \cos\theta, \sin\theta, \varepsilon) \tag{5.6b}$$

with $|f_1(J)|$, $|f_2(J)| \leqslant K_3\varepsilon$.

By some standard theorems of ordinary differential equations, we get the existence of a unique solution of the form

$$\theta(J) = \theta(s) + \frac{1}{\varepsilon}\int_s^J [\mathrm{Im}\,\lambda_1(\tau) + \varepsilon K_4(s, J)]\, d\tau \tag{5.7}$$

for the initial value problem of (5.6b) from $J=s$, where $|K_4(s,J)| \leqslant c_4$ is uniformly bounded. At $J=J_-$,

$$\theta(J_-)=\theta(s)+\frac{1}{\varepsilon}\int_s^{J_-}[\operatorname{Im}\lambda_1(\tau)+\varepsilon K_4(s,J_-)]\,d\tau. \tag{5.8}$$

From this and (5.6a), it follows that

$$\varepsilon R_J=(\operatorname{Re}\lambda_1(J)+f_1)\,R. \tag{5.9}$$

Then

$$R(J_-)=R(s)\,e^{(1/\varepsilon)(\int_s^{J_-}[\operatorname{Im}\lambda_1(\tau)+\varepsilon K_4(s,J_-)]\,d\tau)}, \tag{5.10}$$

where $|K_5(s,J_-)| \leqslant c_5$.

Since all the coefficients in the equation are polynomials in J and λ_1, and we have nice initial conditions, the resulting semigroup is analytic. Indeed, we see that $R(s,J)$, $\theta(s,J)$ are analytic functions of s and J as long as λ_1 is analytic.

We see from (5.8) and (5.10) that

$$\begin{aligned} Z(J_-)&=R(J_-)\,e^{i\theta(J_-)}\\ &=Z(s)\,e^{(1/\varepsilon)(\int_s^{J_-}[\lambda_1(\tau)+\varepsilon F_2(s)]\,d\tau)}, \end{aligned} \tag{5.11}$$

where $|F_2(s)| \leqslant c_6$.

From $\binom{V_s(s,s)}{W_s(s,s)}=\binom{1}{\gamma-1}$, we obtain

$$\begin{pmatrix} Z(s)\\ \bar{Z}(s)\end{pmatrix}=A_1^{-1}(s)\begin{pmatrix}1\\ \gamma-1\end{pmatrix}. \tag{5.12}$$

If we write $A_1^{-1}=\binom{\eta_1}{\eta_2}$, then $Z(s)=\eta_1\binom{1}{\gamma-1}$.

$$\begin{aligned} \begin{pmatrix} V_s(J_-)\\ W_s(J_-)\end{pmatrix}&=Z(J_-)\,\xi_1(J_-)+\bar{Z}(J_-)\,\bar{\xi}_1(J_-)\\ &=2\operatorname{Re}\left\{\eta_1\begin{pmatrix}1\\ \gamma-1\end{pmatrix}e^{(1/\varepsilon)(\int_s^{J_-}[\lambda_1(\tau)+\varepsilon F_2(s)]\,d\tau)}\,\xi_1(J_-)\right\}. \end{aligned} \tag{5.13}$$

To estimate the distance at the center plane $J=J_-$ between two solutions, one of which starts on the frame solution at $s_1<J_-$, and the other at $s_2>J_-$, we look at

$$\begin{pmatrix} V(J_-, s_2) \\ W(J_-, s_2) \end{pmatrix} - \begin{pmatrix} V(J_-, s_1) \\ W(J_-, s_1) \end{pmatrix}$$

$$= \int_{s_1}^{s_2} \begin{pmatrix} V_s(J_-, s) \\ W_s(J_-, s) \end{pmatrix} ds$$

$$= 2\mathrm{Re} \left\{ \int_{s_1}^{s_2} \eta_1(s) \begin{pmatrix} 1 \\ \gamma^{-1} \end{pmatrix} e^{(1/\varepsilon)(\int_s^{J_-} [\lambda_1(\tau) + \varepsilon F_2(s)]\, d\tau)} \xi_1(J_-)\, ds \right\}. \tag{5.14}$$

As we noticed before, all integrands are real analytic functions of s. Hence we are able to extend the solution to the complex plane, and to apply contour integration.

6. λ IN THE COMPLEX DOMAIN

In order to show that the right-hand side of (5.14) is exponentially small, we need to look at the complex analytic extension of the solution. In this section, we describe the global structure of $\lambda_1(z)$ in the z-plane. To accomplish this, we use the specific properties of $\lambda_1(z)$ in the FitzHugh Nagumo System.

Let $\lambda(z)$ be the analytic continuation to the complex z-plane of the function λ_1 defined by (2.3). We first formally write

$$\lambda(z) = \tfrac{1}{2}(3z^2 + \gamma_1)(-3z^2 + 3J_-^2 + \sqrt{(3z^2 - 3J_-^2 - 2b\gamma)^2 - 4b}), \tag{6.1}$$

where J_- and γ_1 are defined before.

In order to find the branch points for the function $\lambda(z)$, we solve the fourth order equation

$$(3z^2 - 3J_-^2 - 2b\gamma)^2 - 4b = 0.$$

The four roots are

$$J_T^1 = -\tfrac{1}{3}\sqrt{(a^2 - a + 1) + 3b\gamma + 6\sqrt{b}},$$
$$J_T^2 = -\tfrac{1}{3}\sqrt{(a^2 - a + 1) + 3b\gamma - 6\sqrt{b}},$$
$$J_T^3 = \tfrac{1}{3}\sqrt{(a^2 - a + 1) + 3b\gamma - 6\sqrt{b}},$$
$$J_T^4 = \tfrac{1}{3}\sqrt{(a^2 - a + 1) + 3b\gamma + 6\sqrt{b}}.$$

J_T^1 and J_T^4 are always real, and $J_T^1 = -J_T^4 < J_- < J_+ < J_T^4$. $J_T^2 = -J_T^3$ is real whenever $(a^2 - a + 1) + 3b\gamma - 6\sqrt{b} \geq 0$, and $J_T^2 = -J_T^3$ is pure imaginary whenever $(a^2 - a + 1) + 3b\gamma - 6\sqrt{b} < 0$.

Thus depending on the sign of $(a^2-a+1)+3b\gamma-6\sqrt{b}$, $\lambda(z)$ can have two real branch points, and two imaginary branch points, or four real branch points. We denote the former one as Case I, the latter one as Case II.

In Case I, J_T^1 and J_T^4 are the turning points on the real axis where the eigenvectors coincide, and the branch points J_T^2 and J_T^3 are on the imaginary axis.

In Case II, all J_T^1, J_T^2, J_T^3, J_T^4 are turning points on the real axis.

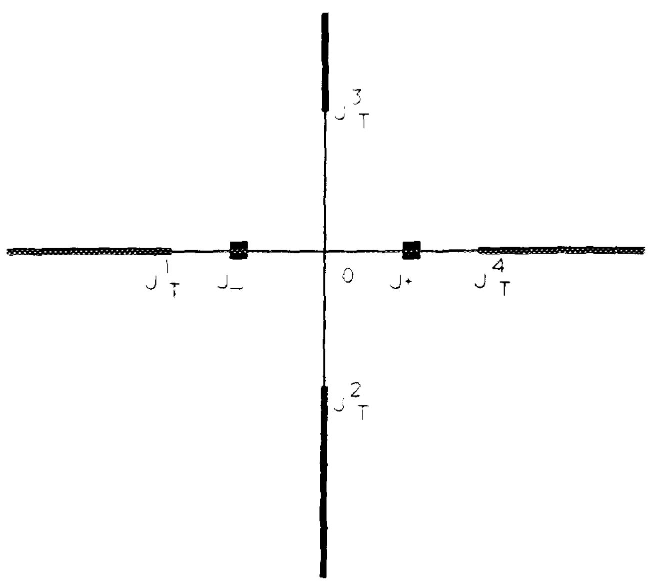
We describe the properties of $\lambda(z)$ according to the following steps:

(1) Since $\lambda(z)$ contains a square root sign, we use the standard techniques of complex analysis to cut the z-plane into a domain where $\lambda(z)$ is analytic.

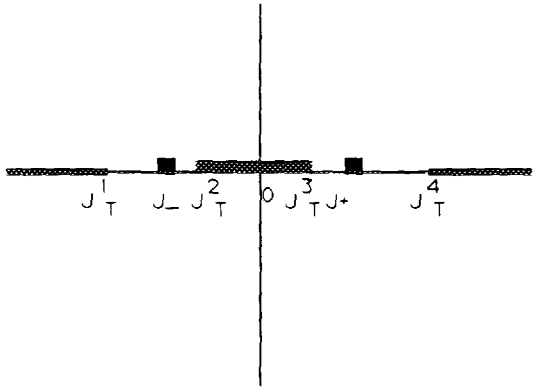
In Case I, we make cuts from J_T^1 to $-\infty$, from J_T^2 to $-i\infty$, J_T^3 to $+i\infty$, and J_T^4 to $+\infty$ shown in Graph A. In Case II, we make cuts from J_T^1 to $-\infty$, from J_T^2 to J_T^3, and from J_T^4 to $+\infty$ as shown in Graph B.

We just look at Case I from now on, and we mention the result in Case II later. Since we are interested in the lower half plane, we just state the results in the lower half plane. Results in the upper half plane can also be obtained through the same methods.

The function $\lambda(z)$ is determined in the cut plane by the choice that $\operatorname{Im} \lambda(J_-) > 0$.



GRAPH A



GRAPH B

Along the cut from J_T^1 to $-\infty$ in the third quadrant (the lower part of the negative real axis),

$$\lambda(z)=\tfrac{1}{2}(3z^2+\gamma_1)(-3z^2+3J_-^2+|(3z^2-3J_-^2-2b\gamma)^2-4b|^{1/2}), \quad (6.2)$$

where the square root is defined as the positive root of the positive argument.

Moreover, $\lambda(z)$ in the third quadrant can be obtained by the continuation of $\lambda(z)$ from the cuts where $\lambda(z)$ is known. In particular, we would like to know the asymptotic properties of $\lambda(z)$ in the third quadrant.

We define the new domain D_3 to be the z-plane with the two cuts, one of which lies in the second quadrant and goes from J_T^1 to J_T^3, and the other lies in the fourth quadrant and goes from J_T^2 to J_T^4. Let $L_3(z)$ be the analytic function of z in the domain D_3 such that

$$L_3(z)=\lambda(z) \quad (6.3)$$

when z is in the third quadrant.

In the domain D_3, we observe that $L_3(z)$ is analytic in the neighborhood of ∞. From (6.2), we see that when $z\leqslant J_T^1$ is on the real axis, $3z^2-3J_-^2-2\beta\gamma>0$ so that

$$L_3(z)=(-4b)(3z^2+\gamma_1)-2b+O\left(\frac{1}{z^2}\right). \quad (6.4)$$

By the uniqueness of the Laurant expansion of $L_3(z)$, (6.4) is valid for all z in the domain D_3. In particular, since $L_3(z) = \lambda(z)$ when z is in the third quadrant, we see that

$$\lambda(z) = (-4b)(3z^2 + \gamma_1) - 2b + O\left(\frac{1}{z^2}\right) \tag{6.5}$$

when z is in the third quadrant and $|z|$ is large enough.

Similarly, when $|z|$ is large and z is in the fourth quadrant,

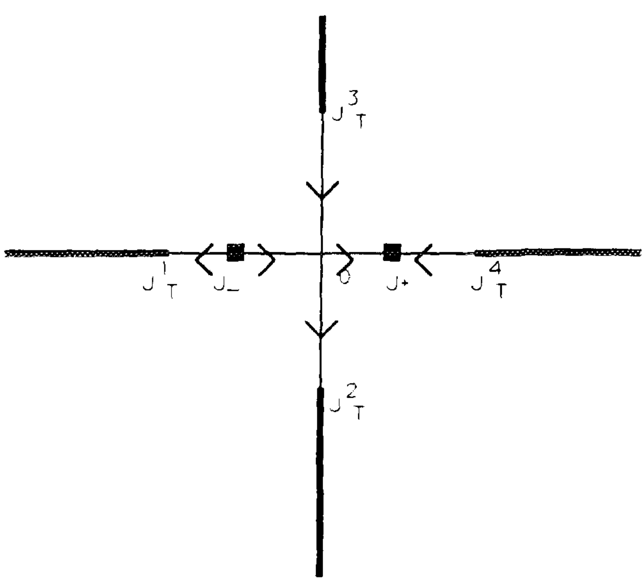
$$\lambda(z) = -(3z^2 - 3J_-^2 - b\gamma)^2 - \frac{1}{\gamma}(3z^2 - 3J_-^2 - b\gamma) - b + O\left(\frac{1}{z^2}\right). \tag{6.6}$$

(2) We define the function

$$\phi(z) = \operatorname{Re} \int_0^z \lambda(\tau)\, d\tau.$$

We consider $\phi(z)$ along the real and imaginary axis first.

We notice that $\phi(z) = -\phi(-z)$ when z is real. If z is real and $z < J_-$, then $\phi(z)$ is strictly decreasing in z. If z is real and $J_- < z < J_+$, then $\phi(z)$ is strictly increasing as z goes from J_- to J_+. For z real and $z \geqslant J_+$, $\phi(z)$ decreases.



GRAPH C

Between the branch points J_T^2 and J_T^3 on the imaginary axis, $\operatorname{Im} \lambda(z) > 0$. Then ϕ is strictly decreasing as z goes upwards. Along the branch cut on the imaginary axis from J_T^2 to $-i\infty$, $\operatorname{Im} \lambda(z)$ is continuous, and hence $\phi(z)$ is continuous across the cut. Moreover since $\operatorname{Im} \lambda(z) = 0$, ϕ is a constant on the vertical cut. We use Graph C to indicate the directions in which ϕ increases.

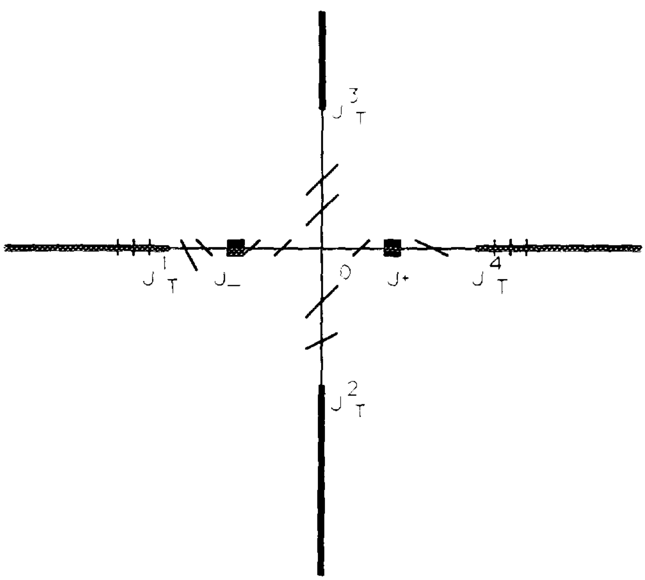
We observe that $\phi(z)$ jumps across the horizontal cuts, and $\phi'(z)$ jumps on the vertical cut.

(3) If we write any curve in the z-plane as $z = z(\sigma)$ where σ is a real parameter, then $\phi(z(\sigma))$ is increasing whenever $(d/d\sigma)\,\phi \geqslant 0$. A necessary and sufficient condition for $z = z(\sigma)$ to be the level curve of ϕ is $\operatorname{Re}(\lambda(z(\sigma))(dz/d\sigma)) = 0$.

We examine the directions of the level curves of ϕ through the points of the real and imaginary axes. See Graph D.

At a point of the z-plane where $\lambda \neq 0$, if one lets $z = z_1 + iz_2$, then one obtains the direction of the tangent to the level curve from the relationship

$$\frac{\partial \phi}{\partial z_1} - i\,\frac{\partial \phi}{\partial z_2} = \lambda(z).$$



GRAPH D

We see that if $\lambda \neq 0$, then the level curve is uniquely determined by its initial point.

LEMMA 1. *Let Ω be the region in the cut z-plane, and suppose that $\lambda(z) \neq 0$ for all $z \in \Omega$. Then given any $z_0 \in \Omega$, there exists a unique curve $z = z(\sigma)$ s.t. $\phi(z(\sigma)) = \phi(z_0)$.*

Remark. Zeros of λ are the critical points for the ϕ level curves, since at such points the level curves can not be traced by the differential equation. However, the local behavior of ϕ near the zeros of $\lambda(z)$ can be obtained by asymptotic expansions. From (6.1), $\lambda(z) = 0$ if and only if $3z^2 + \gamma_1 = 0$. Thus $z(+) = i\sqrt{\gamma_1/3}$ and $z(-) = -i\sqrt{\gamma_1/3}$ are the only two zeros of $\lambda(z)$. $z(-)$ and $z(+)$ are located on the cuts from J_T^2 to $-i\infty$ and J_T^3 to $i\infty$, respectively. Besides those cuts themselves are level curves, one can show that other level curves also come into these points from both sides of the cut.

(4) We notice two properties of the level curves in the cut z-plane.

(a) A level curve can not return to its starting point. Otherwise, the level curve will form a loop. Since $\phi(z) = \operatorname{Re} \int_{J_-}^{z} \lambda(\tau)\, d\tau$ is harmonic, and also is constant along the loop, this would imply that ϕ is constant so that $\lambda(z) = 0$ inside the loop. This contradicts the fact that $z(-)$ and $z(+)$ are the only zeros of λ.

(b) A level curve can not end at a finite point z_0 unless $\lambda(z_0) = 0$.

We also need to study the asymptotic behavior of ϕ in the fourth quadrant and the third quadrant.

In the fourth quadrant, in the neighborhood of ∞, $\lambda(z)$ has the asymptotic expansion expressed in (6.6). If we write $z = z_1 + iz_2$, then the equation for the level curve $\phi(z) = c$ becomes

$$\tfrac{9}{5}(z_1^5 - 5z_1^3 z_2^2 + 5z_1 z_2^4) + C_4(z_1^3 - 3_1^2 z_2) + C_2(z_1^2 - z_2^2) + C_1 z_1 + O(1) = C. \tag{6.7}$$

We consider the asymptotic behavior of the level curve as $|z| \to \infty$. Either z_1 stays bounded, which leads to $z_1 \to 0$, or $z_1 \to \infty$, $z_2 \to \infty$. In the latter case z_1/z_2 will tend to a root of the limiting equation $r^4 - 5r^2 + 5 = 0$. Since the asymptotes are the fourth quadrant, the possible directions will be $r_i = -\sqrt{5/2 + (-1)(i\sqrt{5}/2)}$. Hence in this quadrant, there are three asymptotic directions.

We construct the neighborhood of the imaginary axis N_1: $N_1 = \{z: z = z_1 + iz_2,\ 0 \leqslant z_1 \leqslant 1,\ |z_2| \text{ is large}\}$, and the neighborhoods

$$N_2 = \left\{z: z = z_1 + iz_2, \left|\frac{z_1}{z_2} - r_1\right| \leqslant \frac{1}{8}|r_1|, |z_2| \text{ is large}\right\}$$

$$N_3 = \left\{z: z = z_1 + iz_2, \left|\frac{z_1}{z_2} - r_2\right| \leqslant \frac{1}{8}|r_2|, |z_2| \text{ is large}\right\}$$

of the other two asymptotic lines.

Since

$$\frac{d\phi}{dz_1} = 9(z_1^4 - 6z_1^2 z_2^2 + z_2^4) + O(z_1^2, z_2^2) \tag{6.8}$$

in the neighborhood of ∞, we find that

$$\begin{aligned} &\frac{d\phi}{dz_1} > 9z_2^4 + O(z_2^2) > 0 && \text{in } N_1 \\ &\frac{d\phi}{dz_1} < -\frac{1}{9}z_2^4 + O(z_2^2) < 0 && \text{in } N_2 \\ &\frac{d\phi}{dz_1} < -\frac{1}{9}z_2^4 + O(z_2^2) < 0 && \text{in } N_3. \end{aligned} \tag{6.9}$$

Because of the monotonicity property of ϕ in the horizontal direction within each of the neighborhoods N_1, N_2, and N_3, any horizontal line can intersect one level curve with the value ϕ at most, and only intersect at most once with it in the same neighborhood.

In the third quadrant, the situation is even simpler. From (6.5),

$$\phi(z) = \text{Re}(-4b)(z^3 + \gamma_1 z) - 2bz + O(1), \tag{6.10}$$

when z is in the third quadrant.

There are at most two directions to which the level curves could approach asymptotically, one of which is $z_1/z_2 = \sqrt{3}$ and the other one is the cut.

We can also show that in the designated neighborhood of each asymptote, any horizontal line only can intersect one level curve of level ϕ once at most.

By combining the results which we showed, we can construct for any prescribed $K > 0$ a half ball $B = \{z: |z| \leqslant R = R(K),\ \text{Im}\, z \leqslant 0\}$ such that if a level curve with a level $|\phi| \leqslant K$ exits B at a point z with $\text{Im}\, z < 0$, then it must approach ∞ within the neighborhood of one of the asymptotes without coming back into B.

(5) We look at the level curves in the neighborhood of J_- and J_+ first.

We noticed that along the real axis, ϕ has a minimum at J_- and a maximum at J_+, and $\operatorname{Im} \lambda(z) > 0$ when $z = J_-$ or $z = J_+$. Since

$$\frac{\partial \phi}{\partial z_2} = -\operatorname{Im} \lambda(z) \tag{6.11}$$

which is negative when $z = J_-$, there are level curves of ϕ in the lower half plane which connect points on the real axis near and below J_- with points of the real axis near and above J_-.

It is possible that if we start the level curve from a point too far to the left of J_-, it no longer reaches a point of the real axis to the right of J_-. Let J_a be the infimum of real J such that the level curve in the lower half plane which starts at J returns to the real axis. Since all the connected level curves are inside B which is compact, the level curve in the lower half plane through J_a must have one of two properties: (1) it hits the vertical cut (hence it hits $z(-)$ by the remark we made at the end of (3)), or (2) it intersects the real axis tangentially at a point to the right of J_-.

If we study cases (1) and (2) more carefully, we find that if $\phi(z(-)) < \phi(J_+)$, then (1) is true, and if $\phi(z(-)) > \phi(J_+)$, (2) happens.

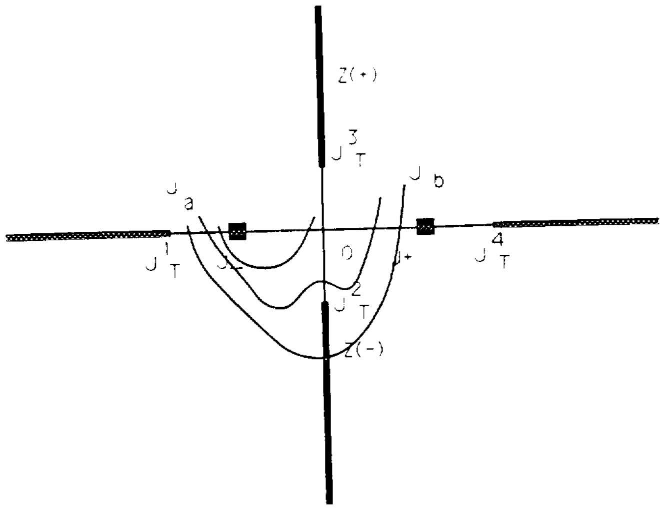
If (1) happens, then the level curve starting from J_a which can be treated as the limit of level curves consists of an arc from J_a to $z(-)$, a part of the cut, and one arc from $z(-)$ to some $J_b > J_-$.

If (2) happens, then the level curve Γ_{J_a} connects J_a with J_+ from the lower half plane since $J_+ > J_-$ is the only other point on the real axis such that $\operatorname{Re} \lambda(z) = 0$.

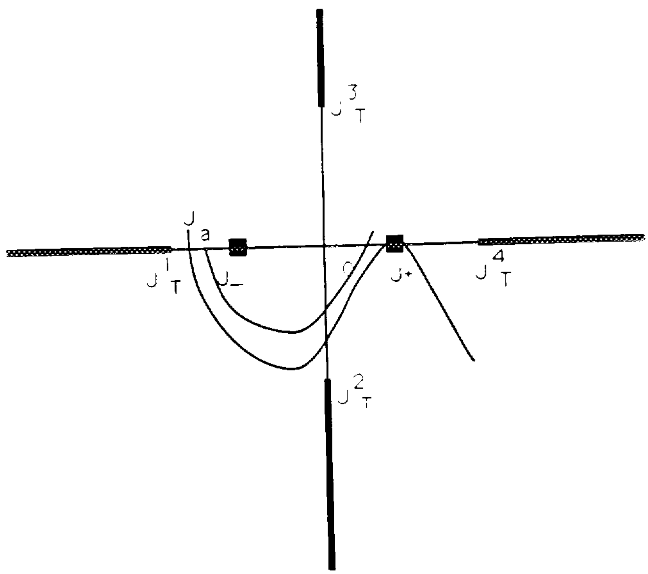
We summarize these results in the following two lemmas.

LEMMA 2. *We compare the values of $\phi(z(-))$ and $\phi(J_+)$. If $\phi(z(-)) \leqslant \phi(J_+)$, then in the cut z-plane, there exists a ϕ-level curve $L_{z(-)}$ connecting $z(-)$ with the real axis at points J_a, J_b such that $J_a < J_- < J_b$, and $\phi(J_a) = \phi(J_b) = \phi(z(-))$. On the other hand if $\phi(z(-)) > \phi(J_+)$, then there is a level curve L_{J_+} which connects J_+ with $J_a < J_-$ on the real axis.*

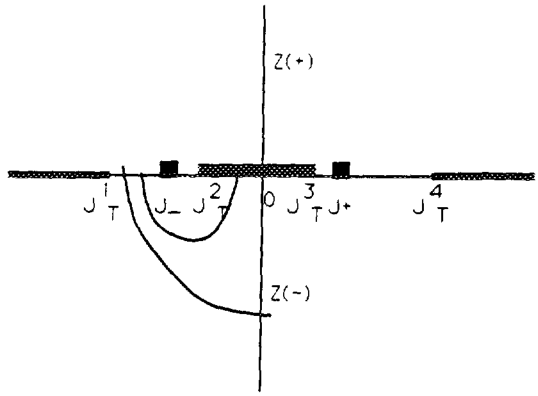
LEMMA 3. *For any $J < J_-$ with $\phi(J) < \min(\phi(J_+), \phi(z(-)))$, there exists a level curve L_J of ϕ in the lower half of the plane which connects J with another point $J^* > J_-$ on the real axis. When $\phi(J) > \min(\phi(z(-)), \phi(J_+))$ there is no such curve.*



GRAPH E



GRAPH F



GRAPH G

Remark. Both $\phi(z(-)) > \phi(J_+)$ and $\phi(z(-)) < \phi(J_+)$ happen for some ranges of parameters. Two simple examples are (1) when we adjust $a^2 - a + 1 - 3b\gamma$ to be close to 0, so that $J_+ - J_-$ is close to 0, we can show that $\phi(z(-)) > \phi(J_+)$; (2) when $2\sqrt{b} - 2b\gamma$ is small and close to $\frac{1}{3}(a^2 - a + 1 - 3b\gamma)$, we can show that $\phi(z(-)) < \phi(J_+)$.

Graph E corresponds to the case where $\phi(z(-)) < \phi(J_+)$, and Graph F for $\phi(z(-)) > \phi(J_+)$.

In Case II, there are four real turning points, and we use the same strategies to treat the problem. We notice that, in this situation, the imaginary axis is a level curve, and $\phi(z(+)) = \phi(z(-)) = \phi(0) < \phi(J_+)$. Lemmas 2 and 3 remain valid in this case, and are proved in the same way.

Graph G corresponds to Case II.

7. DELAY OF OSCILLATIONS

We resume the study of the distance between two solutions from different sides of the first critical point $J = J_-$. In order to obtain the desired results, we analyze (5.14) by using contour integration. In the various cases, we define the points J_l and J_r to be the pair of points on the real axis which are connected by a level curve of ϕ and have a greatest possible distance. However, to avoid complications caused by the branch points of λ where the eigenvectors collapse so that η_1 might get unbounded, we have to choose J_l and J_r properly so that the level curve is kept away from the cuts. We take $J_l \in (J_T^1, J_-)$ with $\phi(J_l) \leqslant \min[\phi(z(-)), \phi(J_+)]$, and $J_r = J_l^* > J_-$. We write U_- for $\binom{V(J, J_l)}{W(J, J_l)}$, U_+ for $\binom{V(J, J_r)}{W(J, J_r)}$.

From (5.14), we see that

$$|U_+(J_-)-U_-(J_-)| \leqslant 2\left|\int_{J_l}^{J_r} \eta_1(s)\begin{pmatrix}1\\ \gamma^{-1}\end{pmatrix} e^{(1/\varepsilon)(\int_s^{J_-} \lambda_1(J)\,dJ+\varepsilon F_2(s))}\,\xi_1(J_-)\,ds\right|$$

$$=2\left|\int_{L_{J_l}} \eta_1(z)\begin{pmatrix}1\\ \gamma^{-1}\end{pmatrix} e^{(1/\varepsilon)(\int_z^{J_-} \lambda(J)\,dJ+\varepsilon F_2(z))}\,\xi_1(J_-)\,dz\right|, \tag{7.1}$$

where L_{J_l} is the ϕ-level curve connecting the two real points J_l, J_r.

In the z-plane, the $F_2(z)$ are the analytic extensions of these functions on the real axis. Since all these functions can be explicitly expressed in the variable J when they are functions of the real axis, the extensions can be easily obtained, and the functions themselves are bounded. $\eta_1(z)=O(1)$, $F_2(z)=O(1)$.

On L_{J_l}, we see that $\phi(z)=\phi(J_l)=\phi(J_r)$ is a positive constant.

Thus from (7.1),

$$|U_+(J_-)-U_-(J_-)| \leqslant K\int_{L_{J_l}} |\eta_1(z)|\, e^{(1/\varepsilon)(\mathrm{Re}\int_z^0 \lambda(\tau)\,d\tau+\varepsilon F_2(z))}\,d|z|$$

$$\leqslant K_1 e^{-(1/\varepsilon)(\phi(J_l)-\phi(J_-))} \tag{7.2}$$

which is of order $e^{-1/\varepsilon}$.

From Theorem 1, $U_-(J)=O(\varepsilon)$ when $J\leqslant J_-$, and $U_+(J)=O(\varepsilon)$ when $J\geqslant J_-$.

Let

$$J_p=\inf\{J\geqslant J_-,\ |U_- - U_+|=M_2\varepsilon\}, \tag{7.3}$$

where M_2 is a fixed constant chosen to be larger than $|U_-(J_-)-U_+(J_-)|$. Then by the definition, $J_p>J_-$.

Using Theorem 2 we obtain that for $J_-\leqslant J\leqslant J_p$

$$|U_-(J)-U_+(J)| \leqslant K_a\,|U_-(J_-)-U_+(J_-)|\, e^{(1/\varepsilon)\int_{J_-}^{J} \mathrm{Re}(\lambda_1(\tau))\,d\tau}. \tag{7.4}$$

THEOREM 3. *If U_- is the solution of* (2.2) *with the initial condition* $(V_-(J_l), W_-(J_l))=(0,0)$, *$U_+$ is the solution of* (2.2) *with the initial condition* $(V_+(J_r), W_+(J_r))=(0,0)$, *then the solution $U_-(J)$ satisfies $|U_-(J)-U_+(J)|\leqslant M_2\varepsilon$ for the interval $J_-\leqslant J\leqslant J_r-K_4\varepsilon\,|\ln\varepsilon|$ where $K_4=K_4(M_2)$ is a constant and ε is small enough.*

Proof. From the estimate (7.2) and (7.4), we see that

$$|U_-(J) - U_+(J)| \leqslant K_b e^{(1/\varepsilon)\int_{J_l}^{J} \mathrm{Re}(\lambda_1(\tau))\, d\tau} \tag{7.5}$$

provided $J_- \leqslant J \leqslant J_p$.

Since $\int_{J_l}^{J_r} \mathrm{Re}(\lambda_1(\tau))\, d\tau = 0$ and $\mathrm{Re}\,\lambda_1 > 0$ for $J > J_-$,

$$\int_{J_l}^{J_r - K_4\varepsilon\,|\ln \varepsilon|} \mathrm{Re}(\lambda_1(\tau))\, d\tau \leqslant \varepsilon \ln\left(\frac{1}{K_b} M_2 \varepsilon\right) \tag{7.6}$$

if K_4 is large enough. Therefore if $J_- \leqslant J \leqslant J_r - K_4\varepsilon\,|\ln \varepsilon|$, then

$$|U_-(J) - U_+(J)| \leqslant K_b e^{(1/\varepsilon)\int_{J_r}^{J} \mathrm{Re}(\lambda_1(\tau))\, d\tau} \leqslant K_b e^{(1/\varepsilon)\int_{J_l}^{J_r - K_4\varepsilon\,|\ln \varepsilon|} \mathrm{Re}(\lambda_1(\tau))\, d\tau} \leqslant M_2 \varepsilon. \tag{7.7}$$

Remark 1. Since $U_+ \sim \varepsilon$ for $J_- \leqslant J \leqslant J_r - K_4\varepsilon\,|\ln \varepsilon|$, Theorem 2 indicates that $U_- \sim \varepsilon$ for $J_- \leqslant J \leqslant J_r - K_4\varepsilon\,|\ln \varepsilon|$.

Remark 2. Suppose the parameters are in the range so that Case I happens and $\phi(z(-)) > \phi(J_+)$, we see that $U_- = O(\varepsilon)$ for $J_a \leqslant J \leqslant J_+ - K_4\varepsilon\,|\ln \varepsilon|$. If we consider in addition that when $J_+ - K_4\varepsilon\,|\ln \varepsilon| \leqslant J \leqslant J_+$, $0 \leqslant \lambda_i(J) \leqslant K_5\varepsilon\,|\ln \varepsilon|$, then we can show that $U_- = O(\varepsilon)$ for $J \leqslant J_+$. Further, in a similar way to Theorem 1, we can show that $U_- = O(\varepsilon)$ for $J > J_+$ as well since $\mathrm{Re}\,\lambda(J) < 0$ when $J > J_+$. Therefore we find that U_- never jumps.

COROLLARY 3. *A family of solutions $(V, W) = (V(J, \varepsilon), W(J, \varepsilon))$ of (2.2) with the initial conditions which satisfy the property $|V(J_l)| + |W(J_l)| \leqslant M_1\varepsilon$ satisfies $|V(J)| + |W(J)| \leqslant \hat{M}_2\varepsilon$ for $J_l \leqslant J \leqslant J_r - K_4\varepsilon\,|\ln \varepsilon|$ where $\hat{M}_2 = \hat{M}_2(M_1)$ and $\varepsilon \leqslant \varepsilon_0(M_1)$.*

Proof. We see from the theorem above that $U_- \sim O(\varepsilon)$ for $J_l \leqslant J \leqslant J_r - K_4\varepsilon$. Let $(V_A, W_A) = U_-$, $(V_B, W_B) = (V, W)$, and we use Theorem 2 to obtain that

$$|(V, W)(J) - U_-(J)| \leqslant M_3 e^{(1/\varepsilon)\int_{J_l}^{J} \mathrm{Re}(\lambda_1(\tau))\, d\tau}\, |(V, W)(J_l) - U_-(J_l)|.$$

Corollary 3 follows.

THEOREM 4. *If $(V, W) = (v(J, \varepsilon), W(J, \varepsilon))$ is a family of the solutions of (2.2) with initial conditions which satisfy $|(V(J_0), W(J_0))| \leqslant M_1\varepsilon$ for $J_0 \leqslant J_l \leqslant J_-$, then $|(V(J), W(J))| \leqslant \tilde{M}_2\varepsilon$ when $J_0 \leqslant J \leqslant J_r - K_4\varepsilon\,|\ln \varepsilon|$ if $\tilde{M}_2 = \tilde{M}_2(M_1)$ and $\varepsilon \leqslant \varepsilon_0(M_1)$.*

Proof. We see from Theorem 1 that $|(V, W)| \leqslant M_2\varepsilon$ when $J_0 \leqslant J \leqslant J_l$. Then Corollary 3 gives Theorem 4.

8. Memory Effects and Jumps

We continue to study the behavior of the solution of (2.2) with the initial conditions

$$V(J_1)=0; \qquad W(J_1)=0 \tag{8.1}$$

for $J_l + K_c\varepsilon\,|\ln \varepsilon| < J_1 < J_-$; i.e., J_1 is closer to J_- than J_l.

We denote $(V(J, J_1), W(J, J_1))$ by $U(J)$. We show there exists J_q such that $U = O(\varepsilon)$ when $J_1 \leqslant J \leqslant J_q$, and also that $|U|/\varepsilon$ becomes large when J moves above J_q. Moreover $|J_q - J_1^*| \leqslant c_4\varepsilon$.

Theorem 2 gives

$$\begin{aligned}
&\frac{1}{M_3} e^{(1/\varepsilon)(\phi(J_-)-\phi(J_1))}\,|U(J_1)-U_-(J_1)| \\
&\quad \leqslant |U(J_-)-U_-(J_-)| \\
&\quad \leqslant M_3 e^{(1/\varepsilon)(\phi(J_-)-\phi(J_1))}\,|U(J_1)-U_-(J_1)|.
\end{aligned} \tag{8.2}$$

Thus using (8.2) and (7.2), we get

$$\begin{aligned}
|U(J_-)-U_+(J_-)| &\leqslant |U(J_-)-U_-(J_-)| + |U_+(J_-)-U_-(J_-)| \\
&\leqslant M_3 e^{(1/\varepsilon)(\phi(J_-)-\phi(J_1))}\,|U(J_1)-U_-(J_1)| \\
&\quad + K_a e^{(1/\varepsilon)(\phi(J_-)-\phi(J_l))};
\end{aligned} \tag{8.3}$$

and also

$$\begin{aligned}
|U(J_-)-U_+(J_-)| &\geqslant |U(J_-)-U_-(J_-)| - |U_+(J_-)-U_-(J_-)| \\
&\geqslant M_3 e^{(1/\varepsilon)(\phi(J_-)-\phi(J_1))}\,|U(J_1)-U_-(J_1)| \\
&\quad - K_a e^{(1/\varepsilon)(\phi(J_-)-\phi(J_l))}.
\end{aligned} \tag{8.4}$$

As stated in the remark at the end of Section 3, $C_2 > |U_-(J_1)|/\varepsilon > C_1$ where C_i are constants independent of ε. $U(J_1)=0$ from (8.1). Since $J_l + K_c\varepsilon\,|\ln \varepsilon| < J_1 < J_-$, $0 > \phi(J_1) > \phi(J_l) + C(K_c)\varepsilon\,|\ln \varepsilon|$, and hence

$$e^{-(1/\varepsilon)\phi(J_1)} < K_5 \varepsilon e^{-(1/\varepsilon)\phi(J_l)} \tag{8.5}$$

for some constant K_5 when ε is small enough.

Thus from (8.3) and (8.4),

$$\frac{1}{K_6}\varepsilon e^{-(1/\varepsilon)(\phi(J_1)-\phi(J_-))} < |U(J_-) - U_+(J_-)|$$
$$< K_6 \varepsilon e^{-(1/\varepsilon)(\phi(J_1)-\phi(J_-))}, \tag{8.6}$$

where K_6 is a constant.

We define

$$J_q = \inf\{J > J_-, |U(J) - U_+(J)| = M_2 \varepsilon\}. \tag{8.7}$$

We take $M_2 > 4K_6 \varepsilon e^{(1/\varepsilon)(\phi(J_-)-\phi(J_1))}$, so that $J_q > J_-$ by the definition and (8.6).

For $J_- \leqslant J \leqslant J_r$, $U_+ = O(\varepsilon)$. By Theorem 2 and (8.6), we see that when $J_1 \leqslant J \leqslant J_q$,

$$\frac{1}{M_3 K_6}\varepsilon e^{(1/\varepsilon)(\phi(J)-\phi(J_1))} < |U(J) - U_+(J)| < M_3 K_6 \varepsilon e^{(1/\varepsilon)(\phi(J)-\phi(J_1))}. \tag{8.8}$$

THEOREM 5. *If U is the solution* (2.2) *initially starting from the frame solution at J_1 such that $J_l + K_c \varepsilon |\ln \varepsilon| < J_1 < J_-$, then $|U(J) - U_+(J)| \leqslant M_2 \varepsilon$ when and only when $J_1 \leqslant J \leqslant J_q$ where $|J_q - J_1^*| = O(\varepsilon)$ and $J_1^* > J_-$ satisfies $\phi(J_1^*) = \phi(J_1)$. In other words, when ε is sufficiently small, there is a J_q such that $J_1^* - c_4\varepsilon \leqslant J_q \leqslant J_1^* + c_4 \varepsilon$ where $c_4 = c_4(M_2)$, such that $|U(J) - U_+(J)| < M_2 \varepsilon$ for $J_1 \leqslant J < J_q$, and $|U(J_q) - U_+(J_q)| = M_2 \varepsilon$.*

Proof. $\phi(J)$ is an increasing function of J for $J > J_-$. There exists c_4 such that $\phi(J_1^* + c_4\varepsilon) > \phi(J_1) + \varepsilon \ln(M_2 K_6 M_3 \varepsilon)$ and $\phi(J_1^* - c_4 \varepsilon) < \phi(J_1) + \varepsilon \ln((M_2/K_6 M_3)\varepsilon)$. Therefore from (8.8), there exists $J_q \in (J_1^* - c_4 \varepsilon, J_1^* + c_4\varepsilon)$ such that $|U(J) - U_+(J)| \leqslant M_2 \varepsilon$ when $J_1 \leqslant J \leqslant J_q$ and $|U(J_q) - U_+(J_q)| = M_2 \varepsilon$.

9. LARGE AMPLITUDE SOLUTIONS

Now we examine the development of solutions when J passes through J_q. We first show the existence of large amplitude solutions for $J \geqslant J_-$.

Let $V_4 = V - V_+$, $W_4 = W - W_+$. (V_4, W_4) satisfies the system

$$\varepsilon \frac{d}{dJ} V_4 = (3J^2 + \gamma_1)(-(3J^2 + 3J_-^2 - b\gamma) V_4 - W_4)$$
$$+ h_4(V_4, V_+, J) \tag{9.1a}$$

$$\varepsilon \frac{d}{dJ} W_4 = (3J^2 + \gamma_1)(bV_4 - b\gamma W_4), \tag{9.1b}$$

where $h_4 = F_1(V_+ + V_4, J) - F_1(V_+, J)$ is a polynomial in V_4 of the highest degree three and the lowest degree one.

If $J_- \leqslant J \leqslant J_T^2$ (or J_T^4), then we can write $\binom{V_4}{W_4} = Z(J)\,\xi_1(J) + \bar{Z}(J)\,\bar{\xi}_1(J)$. $Z(J)$ satisfies the equation

$$\varepsilon \frac{d}{dJ} Z = \lambda_1(J)\,Z + H_6(Z, \bar{Z}, J), \tag{9.2}$$

where

$$\begin{aligned} H_6 = &\sum_{p+q=1} A_{p,q}(J)\,Z^p\bar{Z}^q + \sum_{p+q=2} B_{p,q}(J)\,Z^p\bar{Z}^q \\ &+ \sum_{p+q=3} C_{p,q}(J)\,Z^p\bar{Z}^q \end{aligned} \tag{9.2a}$$

with $|A_{p,q}(J)| \leqslant M\varepsilon$.

We use the classical canonical transformation which is usually used to derive the Hopf bifurcation, by making the change of variable

$$Z = Y + \sum_{p+q=2} P_{p,q}(J)\,Y^p\bar{Y}^q + \sum_{p+q=3} Q_{p,q}(J)\,Y^p\bar{Y}^q \tag{9.3}$$

to obtain an equation of the form

$$\varepsilon \frac{d}{dJ} Y = \lambda_1(J)\,Y + \alpha(J)\,|Y|^2\,Y + H_7(Y, \bar{Y}, J), \tag{9.4}$$

where

$$\begin{aligned} H_7 = &\sum_{p+q=1} \hat{A}_{p,q}(J)\,Y^p\bar{Y}^q + \sum_{p+q=2} \hat{B}_{p,q}(J)\,Y^p\bar{Y}^q + \sum_{p+q=3} \hat{C}_{p,q}(J)\,Y^p\bar{Y}^q \\ &+ \sum_{p+q=4} D_{p,q}(J)\,Y^p\bar{Y}^q + \sum_{5 \leqslant p+q \leqslant 9} E_{p,q}(J)\,Y^p\bar{Y}^q \end{aligned} \tag{9.5}$$

such that $|\hat{A}_{p,q}(J)| \leqslant M\varepsilon$, $|\hat{B}_{p,q}(J)| \leqslant M\varepsilon$, $|\hat{C}_{p,q}(J)| \leqslant M\varepsilon$.

From computations, we see that $\operatorname{Re}\alpha(J) < 0$. A complete computation of such a coefficient can be found in [5][8].

The norm of Y satisfies

$$\varepsilon \frac{d}{dJ} |Y| = \operatorname{Re}(\lambda_1(J))\,|Y| + \operatorname{Re}(\alpha(J))\,|Y|^3 + H_8(Y, \bar{Y}, J), \tag{9.6}$$

where H_8 is real, and the coefficients have the same properties as those of H_7.

We perform another change of variable $Y \to \beta(J)X$ to simplify the equation. By choosing a suitable $\beta(J) > 0$, we may assume without loss of generality that $\mathrm{Re}(\alpha(J)) = -1$. Let $L(J) = \mathrm{Re}(\lambda_1(J))$, so that $L(J)$ is an increasing function of J in the neighborhood of $J = J_-$, and $L(J_-) = 0$. We now rewrite (9.6) as

$$\varepsilon \frac{d}{dJ}|X| = L(J)|X| - |X|^3 + H_8(X, \bar{X}, J). \tag{9.7}$$

For $J \in \{J \mid K\varepsilon^{1/2} \leqslant L(J) \leqslant \sigma\}$ where σ is small to be fixed later, we construct two surfaces

$$S_1 = \{X, |X| = \sqrt{2L(J)}\}, \tag{9.8a}$$

$$S_2 = \{X, |X| = \sqrt{\tfrac{1}{2}L(J)}\}. \tag{9.8b}$$

We show that the set of points $S_{1,2} = \{X, \sqrt{\frac{1}{2}L(J)} \leqslant |X| \leqslant \sqrt{2L(J)}\}$ bounded by S_1 and S_2 is an invariant set for Eq. (9.7).

On the surface S_1, we see that if σ is sufficiently small and $\varepsilon \leqslant \varepsilon_0(\sigma)$, then

$$\begin{aligned}\varepsilon \frac{d}{dJ}(|X| - \sqrt{2L(J)})\Big|_{|X| = \sqrt{2L(J)}}\\ \leqslant -L(J)^{3/2}\left(1 - \frac{O(\varepsilon)}{L(J)} - O(L(J)^{1/2})\right) - \varepsilon \frac{L'(J)}{2\sqrt{2L(J)}} \leqslant 0. \end{aligned} \tag{9.9}$$

when $K\varepsilon^{1/2} \leqslant L(J) \leqslant \sigma$. Thus the direction of the flow on the surface S_1 is always pointing downwards.

On S_2, we can also get that if σ is sufficiently small and $\varepsilon \leqslant \varepsilon_0(\sigma)$, then

$$\begin{aligned}\varepsilon \frac{d}{dJ}(|X| - \sqrt{\tfrac{1}{2}L(J)})\Big|_{|X| = \sqrt{(1/2)L(J)}}\\ \geqslant \frac{1}{4}L(J)^{3/2}\left(1 - \frac{O(\varepsilon)}{L(J)} - O(L(J)^{1/2})\right) - \varepsilon \frac{2L'(J)}{\sqrt{(1/2)L(J)}}\\ \geqslant \frac{1}{8}L(J)^{3/2} - \varepsilon \frac{C}{L(J)^{1/2}}, \end{aligned} \tag{9.10}$$

when $K\varepsilon^{1/2} \leqslant L(J) \leqslant \sigma$.

Therefore there are some K large enough so that when $L(J) \geqslant K\varepsilon^{1/2}$,

$$\frac{C}{L(J)^{1/2}} \leqslant \frac{1}{8\varepsilon} L(J)^{3/2}.$$

Thus, the right hand side of (9.10) is positive, the direction of the flow on the surface S_2 is always pointing upwards.

Remark. Since $L'(J_-) \neq 0$, we see that if σ and $\varepsilon \leqslant \varepsilon_0(\sigma)$ are sufficiently small, then there are $\hat{\sigma}$ and $\hat{K}$ such that $K\varepsilon^{1/2} \leqslant L(J) \leqslant \sigma$ whenever $J_- + \hat{K}\varepsilon^{1/2} \leqslant J \leqslant J_- + \hat{\sigma}$. Moreover there exists $\hat{\hat{\sigma}}$ such that $L(J) \geqslant \sigma$ whenever $J_- + \hat{\sigma} \leqslant J \leqslant J_+ - \hat{\hat{\sigma}}$.

THEOREM 6. *The set of points $S_{1,2}$ between S_1 and S_2 is invariant for Eq.* (9.7) *when $J_- + \hat{K}\varepsilon^{1/2} \leqslant J \leqslant J_- + \hat{\sigma}$. Hence there exist some solutions between two surfaces S_1 and S_2. Since these solutions are between S_1 and S_2 whose radii depend on J but not on ε when $\hat{K}\varepsilon^{1/2} \leqslant J \leqslant \hat{\sigma}$, the amplitudes of these solutions are much greater than ε, when ε is small.*

Proof. The invariance of the set $S_{1,2}$ has been established above. We take any solution of (9.7) with the initial value at $J = J_- + \hat{K}\varepsilon^{1/2}$ satisfying

$$\sqrt{\tfrac{1}{2} L(J_- + \hat{K}\varepsilon^{1/2})} \leqslant |X(J_- + \hat{K}\varepsilon^{1/2})| \leqslant \sqrt{2L(J_- + \hat{K}\varepsilon^{1/2})}. \qquad (9.11)$$

Because of the invariance of the set, these solutions are the desired solutions. Comparison arguments like these can be found in [14].

Remark. There are of course infinitely many such solutions.

Theorem 6 only covers those J's such that $K\varepsilon^{1/2} \leqslant \lambda(J) \leqslant \sigma$, i.e., $J_- + \hat{K}\varepsilon^{1/2} \leqslant J \leqslant J_- + \hat{\sigma}$. For other J in the unstable region, $J_- + \hat{\sigma} < J < J_+ - \hat{\hat{\sigma}}$ where $L(J) > \sigma$, there is another result similar to Theorem 6.

We see from (9.7) that for all $J_- + \hat{\sigma} < J < J_+ - \hat{\hat{\sigma}}$ such that $L(J) > \sigma$, there exists a small positive $\delta = \delta(\sigma)$ independent of ε such that the flow on the surface $S_3 = \{X, |X| = \delta\}$ is always pointing upwards for such J's.

We also know from the phase plane of FHN that if we draw a big rectangle R containing the intersection of the curves $-f(V) - W + I = 0$ and $V - \gamma W = 0$, then R is invariant. We use the image of R after various transformations as S_4.

THEOREM 7. *There are large amplitude solutions of* (9.7) *between S_3 and S_4, for the region $J_- + \hat{\sigma} \leqslant J \leqslant J_+ - \hat{\hat{\sigma}}$, i.e., for such J's there are solutions of* (9.7) *whose norms are bounded above by a constant, and bounded below by some constant $\delta > 0$ which is independent of ε.*

If we combine the results of Theorem 6 and Theorem 7, we find

COROLLARY 4. *There exist solutions of* (9.7) *whose amplitudes are bounded above and below by the functions of J which are described in Theorem* 6 *and Theorem* 7, *when $J_- + \hat{K}\varepsilon^{1/2} \leqslant J \leqslant J_+ - \hat{\sigma}$.*

10. Convergence to Large Amplitude Solutions

In this section, we consider the evolution of small amplitude solutions towards large amplitude solutions.

For any solution starting from the frame solution at $J_1 > J_i + K_c\varepsilon\,|\ln\varepsilon|$, we know from Theorem 5 that $|X(J_q)| = M_2\varepsilon$ where $|J_q - J_1^*| \leqslant c_*\varepsilon$ and $\phi(J_1^*) = \phi(J_1)$.

We study the continuation of the solution of the equation

$$\varepsilon\frac{d}{dJ}X = \lambda_1(J)\,X - |X|^2\,X + H_7(X, \bar{X}, J) \tag{10.1}$$

for $J > J_q$.

If we express $X(J) = r(J)\,e^{i\theta(J)}$, then the equations are

$$\varepsilon\frac{d}{dJ}r = L(J)\,r - r^3 + rH_9(r, \theta, J) \tag{10.2a}$$

$$\varepsilon\frac{d}{dJ}\theta = \operatorname{Im}\lambda_1(J) + \operatorname{Im}\beta(J)\,r^2 + H_{10}(r, \theta, J), \tag{10.2b}$$

where

$$\begin{aligned} H_9(r, \theta, J) = {} & \sum_{p+q=1} \tilde{A}_{p,q}(J)\cos^p(\theta)\sin^q(\theta) \\ & + \sum_{p+q=2} \tilde{B}_{p,q}(J)\cos^p(\theta)\sin^q(\theta)\,r \\ & + \sum_{p+q=3} \tilde{C}_{p,q}(J)\cos^p(\theta)\sin^q(\theta)\,r^2 \\ & + \sum_{p+q=4} \tilde{D}_{p,q}(J)\cos^p(\theta)\sin^q(\theta)\,r^3 \\ & + \sum_{5\leqslant p+q\leqslant 9} \tilde{E}_{p,q}(J)\cos^p(\theta)\sin^q(\theta)\,r^{p+q-1} \end{aligned} \tag{10.3}$$

and $|\tilde{A}_{p,q}(J)| \leqslant M\varepsilon$, $|\tilde{B}_{p,q}(J)| \leqslant M\varepsilon$, $|\tilde{C}_{p,q}(J)| \leqslant M\varepsilon$ by the structure of H_7. H_{10} is similar to H_9.

We first assume that J_1, the point where we prescribe initial values for system (2.2), is close to J_- so that its corresponding $J_q = J_q(J_1)$ satisfies $J_- + \hat{K}\varepsilon^{1/2} < J_q < J_- + \hat{\sigma}$. Then $K\varepsilon^{1/2} < L(J) < \sigma$ when $J_q \leqslant J \leqslant J_- + \hat{\sigma}$.

Now we construct a function $\rho(J)$ satisfying the equation

$$\varepsilon\frac{d}{dJ}\rho = L(J)\,\rho - \rho^3 + \rho H_9(\rho, \theta(J), J) \tag{10.4}$$

with the initial condition

$$\sqrt{\tfrac{1}{2} L(J_- + \hat{K}\varepsilon^{1/2})} \leqslant |\rho(J_- + \hat{K}\varepsilon^{1/2})| \leqslant \sqrt{2L(J_- + \hat{K}\varepsilon^{1/2})}, \qquad (10.5)$$

where the function $\theta(J)$ is defined by (10.2b).

Following the same procedure as in Theorem 6, we find that the set is still an invariant region for (10.4). Therefore we see that $\sqrt{\frac{1}{2} L(J)} \leqslant \rho(J) \leqslant \sqrt{2L(J)}$ when $J_- + \hat{K}\varepsilon^{1/2} \leqslant J \leqslant J_+ + \hat{\sigma}$.

Knowing that $|r(J_q)| = M_2\varepsilon > 0$, we are now able to show that $r(J)$ jumps to the large amplitude function $\rho(J)$. We define $r_1 = r/\rho$. Then (10.2a) becomes

$$\varepsilon \frac{d}{dJ} r_1 = r_1(1 - r_1)\left(\rho^2(r_1 + 1) + \rho \frac{\partial}{\partial r} H_9(\tilde{r}_1 \rho, \theta, J)\right), \qquad (10.6)$$

where $0 \leqslant \tilde{r}_1 \leqslant 1$ by the Mean Value Theorem.

We observe that since $r_1(J_q) = M_2\varepsilon/\rho(J_q) < c\varepsilon^{1/2} < 1$ when ε is small, $r_1 < 1$ provided $(\rho^2(r_1 + 1) + \rho(\partial/\partial r) H_9(\tilde{r}_1 \rho, \theta, J)) > 0$ which is true for $J_q \leqslant J \leqslant J_- + \hat{\sigma}$ by the estimate as follows.

From (10.3), we get

$$\left| \frac{\partial}{\partial r} H_9(\tilde{r}_1 \rho, \theta, J) \right| \leqslant E\varepsilon(1 + \rho + \rho^2) + O(\rho^2). \qquad (10.7)$$

Thus for $J_q \leqslant J \leqslant J_- + \hat{\sigma}$, $L(J_q) < L(J) \leqslant \sigma$. Also $\sqrt{\frac{1}{2} L(J)} \leqslant \rho(J) \leqslant \sqrt{2L(J)} \leqslant \sqrt{2\sigma}$ for $\hat{K}\varepsilon^{1/2} \leqslant J_q \leqslant J \leqslant J_- + \hat{\sigma}$ by the assumption on J_q. We obtain

$$K^{(3)}_{\sigma, J_q} \geqslant \left(\rho^2(r_1 + 1) + \rho \frac{\partial}{\partial r} H_9(\tilde{r}_1 \rho, \theta, J)\right) \geqslant K^{(2)}_{\sigma, J_q} > 0. \qquad (10.8)$$

We compare (10.6) with the solutions r_2, r_3 of the equations

$$\varepsilon \frac{d}{dJ} r_2 = K^{(2)}_{\sigma, J_q} r_2(1 - r_2), \qquad (10.9)$$

$$\varepsilon \frac{d}{dJ} r_3 = K^{(3)}_{\sigma, J_q} r_3(1 - r_3), \qquad (10.10)$$

with initial conditions

$$r_j(J_q) = \frac{M_2\varepsilon}{p(J_q)} \leqslant c\varepsilon^{1/2}, \qquad j = 2, 3.$$

The solutions of (10.9) and (10.10) are

$$r_j(J) = \frac{r_j(J_q)}{r_j(J_q) + (1 - r_j(J_q))\, e^{-K^{(j)}_{\sigma, J_q} \cdot (J - J_q)/\varepsilon}}, \qquad j = 2, 3.$$

Therefore

$$\frac{(1 - M_2\varepsilon/p(J_q))\, e^{-K^{(2)}_{\sigma, J_q}(J - J_q)/\varepsilon}}{M_2\varepsilon/p(J_q) + (1 - M_2\varepsilon/p(J_q))\, e^{-K^{(2)}_{\sigma, J_q}(J - J_q)/\varepsilon}}$$
$$\leqslant |r_1 - 1| \leqslant \frac{(1 - M_2\varepsilon/p(J_q))\, e^{-K^{(3)}_{\sigma, J_q} \cdot (J - J_q)/\varepsilon}}{M_2\varepsilon/p(J_q) + (1 - M_2\varepsilon/p(J_q))\, e^{-K^{(3)}_{\sigma, J_q} \cdot (J - J_q)/\varepsilon}}. \tag{10.11}$$

We see that not only the instant jumping occurs, but also the solution will go to become a large amplitude solution for $J_q + c_2\varepsilon\, |\ln \varepsilon| < J < J_- + \hat{\sigma}$.

THEOREM 8. *If the solution of the system* (2.2a), (2.2b) *with the initial condition* $(V, W)(J_1) = (0, 0)$ *reaches the boundary of the* ε*-neighborhood at* $J_q = J_q(J_1)$ *so that* $J_- + \hat{K}\varepsilon^{1/2} < J_q < J_- + \hat{\sigma}$, *then this solution will go to a function* $(V_\rho(J), W_\rho(J))$ *whose amplitude is independent of* ε. *The behavior of the solution can be expressed as* (10.11) *for* $J_q \leqslant J \leqslant J_- + \hat{\sigma}$. *The large amplitude solution* (V_ρ, W_ρ) *is located in the region* $S_{1,2}$.

If $J_q = J_q(J_1) \in (J_- + \hat{\sigma}, J_+ - \hat{\hat{\sigma}})$, i.e., for any J_1 such that its corresponding J_q is not in the range covered by Theorem 8, we can still see that $|X|$ jumps away from 0 after J_q, and then stays at a large amplitude region when $J_q + c_2\varepsilon \leqslant J \leqslant J_+ - \hat{\hat{\sigma}}$.

THEOREM 9. *There exists* $\delta = \delta(\sigma) > 0$ *such that, if* $J_q(J_1) \in (J_- + \hat{\sigma}, J_+ - \hat{\hat{\sigma}})$, *then*

$$|U - U_+| \geqslant \min(M_2\varepsilon e^{(1/2\varepsilon)\,\sigma(J - J_q)}, \delta)$$

for $J_q \leqslant J \leqslant J_+ - \hat{\hat{\sigma}}$. *In other words, if* $J_q = J_q(J_1) \in (J_- + \hat{\sigma},\ J_+ - \hat{\hat{\sigma}})$, *then the solution of* (2.2) *starting from the frame solution at* J_1 *jumps away from the frame solution near* J_q, *and keeps away at a distance no less than* δ *for* $J_q + O(\varepsilon\, |\ln \varepsilon|) < J < J_+ - \hat{\hat{\sigma}}$.

Proof. Let δ be small so that $\delta^2 + H_9(\delta) \leqslant \frac{1}{2}\sigma$. From (9.7), if $|X| \in (0, \delta)$, then

$$\varepsilon \frac{d}{dJ} |X| \geqslant \frac{1}{2} \sigma\, |X| \tag{10.12}$$

because $L(J) > \sigma$ and ε is small.

TABLE I

	Case I		Case II
J_i (initial)	(A) $\phi(z(-)) \geqslant \phi(J_+)$	(B) $\phi(z(-)) < \phi(J_+)$	
$< J_l + K\varepsilon$	Oscillations never occur	No oscillation before J_r	No oscillation before J_r
$> J_l + K\varepsilon$	Oscillates near $J_q > J$	Oscillates near $J_q > J_-$	Oscillates near $J_q > J_-$

Integrating this inequality yields the inequality

$$|X(J)| \geqslant \min(M_2 \varepsilon\, e^{(2/\varepsilon)\,\sigma(J - J_q)}, \delta) \tag{10.13}$$

for $J_q \leqslant J \leqslant J_+ - \hat{\hat{\sigma}}$.

Remark. In order to apply Theorem 9 or Theorem 8, we must know the location of J_q where $|X(J_q)| = M_2 \varepsilon$. We are only able to locate J_q when the solution starts from the frame solution at a point J_1 which satisfies $J_l + c_4 \varepsilon\, |\ln \varepsilon| < J_1 < J_-$, and perhaps from a tiny neighborhood of the frame solution at such J_1. However, if we start the solution at any point $J_0 < J_l$ from the frame solution, the behavior of the solution after J_r is still unknown.

11. Conclusions

We summarize the preceding results in Table I where we assume $(V(I_i), W(J_i)) = (0, 0)$, $|J_q - J_i^*| \leqslant K_1 \varepsilon$, and $\phi(J_i) = \phi(J_i^*)$.

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